Finite Element Method

Лекторы:
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Язык:
English

Трудоемкость:
3 з.е.

Образовательная программа:
Computer Modeling of quantum and nanophotonic systems
2

Лекции (ак.час)*	**Практические занятия (ак.час)**	**Лабораторные занятия (ак.час)**
32	32	
***1 академический час = 45 минутам**		

Motivation:

Modern physics and technology problems often require solution of ordinary or partial differential equations (ODE or PDE). It is rare occasion that the solutions can be expressed analytically. Hence, researchers have to have a good command in numerical methods to obtain the result. One class of techniques to solve PDEs is the Finite Element Methods (FEM). There are some very modern commercial and open-source solvers that are based on these methods (e.g., Comsol). Therefore it is beneficial to gain some knowledge in the method theory as well as personal experience in its program implementation, to be able to professionally utilize broad capabilities of other available tools.

Expected course outcome:

Understanding of FEM theory basics, as well as ability to formulate computational algorithm and implement it as a Python program. Knowledge of specific differences between the FEM and the more publically known Finite Difference methods. Experience in solving practical boundary-value problems based on some 2nd order PDEs using the FEM. Understanding of the mesh generation aspect. The course will imply numerous practical assignments.

Содержание курса

Approximate course content:

Структура курса

1. Simple linear two-point boundary value problems in 1D. Overview of some alternative numerical solution techniques.
2. Refreshment on Scientific Python programming. Mesh generation options in Python.
3. Theoretical aspects of the FEM in 1D. Weak problem formulation. Approximation of various types of the boundary conditions. Matrix assembly technique. Account for the essential boundary conditions.
4. FEM theory for the solution of 2D problems. Convergence analysis basics.
5. Solution of the linear algebraic equations resulting from the FEM approximation. Sparse nature of the system and its account. Matrix storage techniques. Sparse matrix system solvers, their implementations in Python SciPy.
6. Solution of transient problems.

Рекомендуемые ресурсы

1. Kong, Siauw, Bayen, Python Programming and Numerical Methods. A guide for Engineers and Scientists, Elsevier, 2021.
2. Whiteley J., Finite Element Methods: A Practical Guide, 2017

Политика оценивания

• Score-to-mark conversion: 55 -> "3"; 75 -> "4"; 95 -> "5"

Mid-term:

• maximum score for the mid-term: 15 points

• mark will depend on the total gained score

- Have at least 6 completed & accepted Assignments before the final exam date
- Pass the final exam

(waivers may apply if you get > 75 points before the exam)

- Maximum score for the exam: 20 points

Тип самостоятельных заданий

For each topic, students must complete assignments and submit a report. There are up to eight assignments per semester. Each assignment report after acceptance: up to 10 or 15 points

Example:

Problem

Solve the following convection-diffusion equation with variable coefficients using FEM:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)\cdot\frac{du}{dx} + r(x)\cdot u = f(x), \qquad x \in (0, \pi).$$

The coefficients and the right-hand-side (source term) of the equation are:

$$p(x) = (1+x)^2, \quad q(x) = 2 + x - x^2, \quad r(x) = \frac{2x-1}{4}, \quad f(x) = ((1+x)^2 + \frac{2x-1}{4})\sin(x) - x(1+x)\cos(x).$$

Find an approximate solution for the following boundary conditions:

$$u(x=0) = 0$$

$$\left(p(x)\frac{du}{dx} + \frac{1}{2}u\right)\Big|_{x=\pi} = \frac{(e^{\pi}-1)(1+\pi)}{e^{\pi}+2\pi+1}$$

The general solution has the form:

$$u(x) = \sqrt{1+x}\left(K_1 + K_2 e^{-x}\right) + \sin(x).$$

For the specified BCs, the constants K_1 и K_2 are:

$$K_1 = 2\frac{\sqrt{1+\pi}}{(1+e^{-\pi}(1+2\pi))}, \qquad K_2 = -K_1$$

1. (1 point) Check correctness of the proposed exact solution. Use sympy module in Python, if necessary.
2. (1 point) Briefly describe the FEM method theory for this problem (borrow material from your prior reports, if relevant).
3. (6 points) Describe the algorithm and implement the computational code (1st order basis functions) to solve this problem numerically.
   1. Account for a general, non-uniform mesh;
   2. Use numerical integration (Gaussian quadratures) to compute the local matrix and the right-hand-side vector components;
   3. Enable algorithmic mechanism allowing for a quick change of the boundary conditions at the ends of the domain (between 1st, 2nd, and 3rd types) and test it changing the right-end boundary condition to zero Neumann (2nd type) condition.
4. (2 points) Conduct numerical studies to check agreement with the theoretical error estimate (as a function of mesh size), for each of the implemented basis function orders

$$\|E\|_{L_2} = \sqrt{\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}} (U(x) - u(x))^2 dx} \approx C \cdot h^p,$$

1. (*2 point) Implement and test the solution using the 2nd order basis functions. Complete the mesh convergence test from item 4.
2. (*3 points) Implement and test the solution using the 3rd order basis functions. Complete the mesh convergence test from item 4.